Itaka Fibrations

Mar 24

Itaka Fibration Theorem: X normal projective variety, L a l.b. s.t. $\kappa(L) > 0$. Then for sufficiently large $k \in \mathbb{N}(L)$, $\varphi_k : X \dashrightarrow Y_k$ are all birationally equivalent to a fixed algebraic fiber space

$$\varphi_\infty : X_\infty \longrightarrow Y_\infty$$

where X_∞, Y_∞ are normal.

Additionally, the Iitaka dim of the restriction of L to a very general fiber is 0.

φ_∞ is the Iitaka fibration associated to L.

Outline of proof:

1.) For $m \in \mathbb{N}(L)$ and $k >> 0$, φ_{km} are birationally equivalent to some fixed alg. fiber space $\Psi_{(m)} : X_{(m)} \to Y_{(m)}$ where $X_{(m)}, Y_{(m)}$ are normal.

2.) We can choose a common birational model for all $\Psi_{(m)}$, $\varphi_\infty : X_\infty \longrightarrow Y_\infty$.

i.e.
$$\begin{array}{ccc} X_\infty & \longrightarrow & X_{(p)} \\ \downarrow & & \downarrow \\ Y_\infty & \longrightarrow & Y_{(p)} \end{array}$$
s.t. horiz. arrows birational.

3.) If $u_\infty : X_\infty \to X$ is the birational map, and F a very general fiber of φ_∞, then $\kappa(u_\infty^* L|_F) = 0$.

Pf of 1.)

Consider $\varphi_m : X \dashrightarrow Y_m$. Via a sequence of blowups of X, we can resolve codim ≥ 2 indeterminacy of φ_m.

i.e. we obtain $u_m : X_{(m)} \to X$ s.t.

$u_m^* L^{\otimes m} = M_m \otimes \mathcal{O}(F_m)$ where M_m is glob. gen, and F_m eff.

s.t. $u_m^* |L^{\otimes m}| = |M_m| + F_m$

and $Bs(u_m^* |L^{\otimes m}|) = F_m$.

Then we have
$$\begin{tikzcd} X_{(m)} \arrow[r, "u_m"] \arrow[dr, "\psi_m"'] & X \arrow[d, dashed, "\varphi_m"] \\ & Y_m \end{tikzcd}$$

where ψ_m is the morphism defined by M_m.

Just as before, ψ_m factors as
$$\begin{tikzcd} X_{(m)} \arrow[d, "\psi_{km}"'] \arrow[dr, "\psi_m"] & \\ Y'_{km} \arrow[r, "\lambda_k"'] & Y_m \end{tikzcd}$$

Where Y'_{km} is the image of the morphism defined by $|M_m^{\otimes k}|$ and λ_k is finite.

Since M_m is globally generated,

semiample fibration theorem $\Rightarrow$ ψ_{km}'s stabilize to a fixed

alg. fiber space $\psi_{(m)} : X_{(m)} \longrightarrow Y_{(m)}$.

Now,
Since $u_m^* L^{\otimes m} = M_m \otimes \mathcal{O}(F_m)$, we have

$$u_m^* L^{\otimes km} = M_m^{\otimes k} \otimes \mathcal{O}(kF_m), \text{ so}$$

$$|M_m^{\otimes k}| \subseteq |u_m^* L^{\otimes km}| = |u_{m*}\mathcal{O} \otimes L^{\otimes km}|$$

But X is normal and u_m is birational, so $u_{m*}\mathcal{O} = \mathcal{O}$

$\Rightarrow$ $|M_m^{\otimes k}|$ linear subsystem of $|L^{\otimes km}|$

Since X is birational to $X_{(m)}$, we can think of $|M_m^{\otimes k}|$ as determining a rational map from X to $Y_{(m)}$, and we get

$$\begin{tikzcd}
X_{(m)} \arrow[d, "\psi_{(m)}"'] & \sim X \arrow[dl, dashed] \arrow[d, "\varphi_{km}"] \\
Y_{(m)} & Y_{km} \arrow[l, "\mu_k"]
\end{tikzcd}$$

where μ_k is generically finite.

But $\mathbb{C}(Y_{(m)})$ is algebraically closed in $\mathbb{C}(X_{(m)}) = \mathbb{C}(X)$ so μ_k is birational.

$\Rightarrow$ φ_{km} is birationally equivalent to $\varphi_{(m)}$. $\square$

Can also characterize Iitaka dim by growth rate of global sections.

<u>Fact</u>: L l.b. on X irreducible, projective, normal. Then as m gets large, $h^0(L^{\otimes m}) \sim m^{\kappa(L)}$

i.e. $\exists$ constants $a, A > 0$ s.t.

$$a m^{\kappa} \leq h^0(L^{\otimes m}) \leq A m^{\kappa} \quad \text{for suff. large } m \in \mathbb{N}(L).$$

<u>Idea of Pf</u>: Use Thm and Asymptotic R-R, which says that if $n = \dim X$, B a divisor

$$\chi(\mathcal{O}(mB)) = \frac{B^n}{n!} m^n + \text{lower deg terms in } m$$